Journal of Sound and Vibration (1999) **228**(1), 51–68
Article No. jsvi.1999.2402, available online at http://www.idealibrary.com on IDEAL®

PERIODIC STRING RESPONSE TO AN IMPACT AND A SUDDENLY APPLIED CONCENTRATED STATIONARY FORCE

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(*Received* 6 *April* 1998, *and in final form* 19 *May* 1999)

An infinite periodic structure unsteady response to a forced excitation is considered. Any forced excitation can be presented as a sequence or a distribution of impulses. The instantaneous impulse is an infinite sum of harmonic forces of the same amplitude and phase, whose frequencies fill the infinite band as follows from the Fourier transformation of the Dirac delta function. The solution to the problem of a periodic structure steady state response to excitation by a harmonic concentrated stationary force is obtained by reducing the problem to a difference equation and used to calculate the unsteady response. The infinite periodic structure consisting of an infinite stretched string, supported by equidistantly spaced identical suspensions, is considered. Each suspension consists of a spring and a dashpot with viscous damping, in parallel. Small transverse oscillations of the string without bending stiffness are considered. In order to exclude from the solution the string slope, which experience a sudden change at the point, where a concentrated force is applied, and so at every suspension point, the boundary problem is solved over the string unloaded span. The string transverse deflection at an arbitrary point of the span as well as its slope are expressed via the deflection values at the beginning and at the end of the span. Then, two neighbour spans are considered together. A suspension reaction that depends on the suspension point deflection is connected with the string slopes to the right and to the left of the point. The connection involves the string transverse deflection at three successive suspension points and represents the second order difference equations. The solution to this equation allows one to express the string deflections at all suspension points via only two, which relate to ends of the span, where the excitation is applied. Consideration of this span leads to calculation of the string steady state oscillations and response to an impact. To give an example of application, the latter is used to calculate the string response to a suddenly applied force.

1. INTRODUCTION

Any force excitation can be presented as a sequence or a distribution of instantaneous impulses. Each impulse is an infinite sum of harmonic forces of the same amplitude and phase, whose frequencies fill the infinite band. This follows

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from the Fourier transformation of the Dirac delta function. Therefore, the problem of a periodic structure steady state response to a harmonic concentrated force may be considered as an initial one in such an approach.

There are several ways to solve the initial problem. In references [1–7], this problem was solved by means of the Fourier transformation and reduced to the calculation of an improper single integral. This way leads to an undesirable consequetive integration to calculate the periodic structure unsteady response. The next way consists of using the transfer matrix method. One regular span of the periodic structure is considered in this method. There is a certain vector, which determines the disturbance of the structure at an arbitrary point of the span. The initial values problem is solved over this span. The solution forms the transfer matrix, which connects two values of the vector that relate to the beginning and to the end of the span. The connection represents a vector-difference equation of the first order. The solution of this equation allows one to express the vectors via two vectors, which relate to ends of the span, where the excitation force is applied. Consideration of this span completes the solution to the entire problem. If the periodic structure is a periodic beam (see references [8–12]), then the four-dimensional vector, which presents, for example, the beam transverse deflection, slope, curvation and shear, can be used.

The infinite stretched string, supported by the equidistantly spaced identical suspensions, is considered in this paper. Each suspension consists of a spring and a dashpot with viscous damping, in parallel. Only small transverse oscillations of the string are taken into account, the string bending stiffness is neglected and so the string slope experiences a sudden change at every point, where a concentrated force is applied, and so at any suspension point. In order to exclude the string slope from the solution, a new method that leads to a scalar difference equation of the second order has been worked out. In accordance with this method, the boundary problem is solved over every string span. The string transverse deflection at an arbitrary point of the span as well as the slope are expressed via the deflection values at the beginning and at the end of the span. Then, two unloaded neighbour spans are considered. A suspension reaction that depends on the suspension point deflection is connected with the string slopes to the right and to the left of the point. The connection involves the string transverse deflection at three successive suspension points and allows one to obtain the second order difference equation. By solving this equation, one expresses the string deflections at all suspension points via only two, which relate to ends of the loaded span. Consideration of the loaded span completes the solution of the steady state problem and, then, leads to calculation of the string response to an impact. To give an example of application, the latter is used to calculate the string unsteady response to a suddenly applied force.

2. STATEMENT OF THE PROBLEM

An infinite string is stretched by a constant tension force f and supported by periodic suspensions with spacing l, identical stiffness k and viscous damping k_1 (see Figure 1). Let variables t and x denote the time and the co-ordinate along the string.

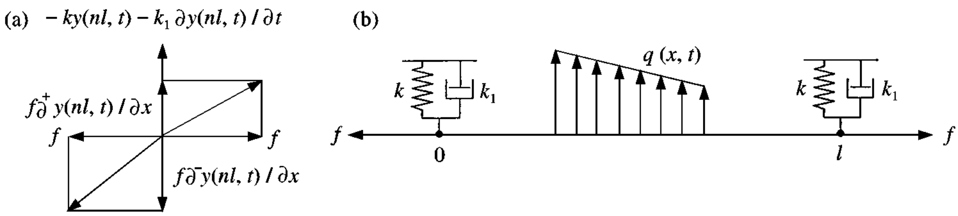

Figure 1. Periodic structure; (a) suspension; (b) span of a string.

The point $x=0$ corresponds to one of suspension points. The small transverse continuous deflection $y(x, t)$ of the string is caused by a transverse load $q(x, t)$ and governed by the linear partial differential equation [13]. A string has no bending stiffness, contrary to a beam. Therefore, its tangent is discontinuous at any point x_0, where a concentrated force is applied. Let $\partial^- y(x_0, t)/\partial x$ and $\partial^+ y(x_0, t)/\partial x$ denote the string tangent slopes to the left and to the right of the point x_0. At the suspension points $x = nl$, where n is an integer, the function $f\partial y(x, t)/\partial x$ experiences the sudden change $ky(nl, t) + k_1\partial y(nl, t)/\partial t$, which is equal to the force, acting upon the suspension (see Figure 1). Therefore,

$$f[\partial^+ y(nl, t)/\partial x - \partial^- y(nl, t)/\partial x] = ky(nl, t) + k_1\partial y(nl, t)/\partial t. \tag{1}$$

Another way to account for the suspension reactions is to add them to the transverse load [14, 15]. In such an approach, the string deflection is governed by the following so-called functional-differential equation [16]:

$$\rho\partial^2 y(x, t)/\partial t^2 - f\partial^2 y(x, t)/\partial x^2 + \sum_{n=-\infty}^{+\infty} [ky(nl, t) + k_1\partial y(nl, t)/\partial t]\delta(x - nl) = q(x, t), \tag{2}$$

where ρ is the string linear density and δ denotes the Dirac function. This equation links the string deflection at an arbitrary point x and those at the suspension points nl.

The string steady state oscillations, caused by the excitation

$$q(x, t) = a_0 \exp(\mathrm{i}\omega_0 t)\delta(x - x_0), \tag{3}$$

were considered in reference [4]. This excitation represents the concentrated harmonic force $a_0 \exp(\mathrm{i}\omega_0 t)$ of the amplitude a_0 and the angular velocity ω_0, which is applied to the point $x_0 = v_0 t$ that moves steadily along the string with the non-zero speed v_0. The solution to problem (2)–(3) was obtained by means of the Fourier transformation in the form of an improper integral. In the particular case of $v_0 = 0$, the solution was yielded in such a form by means of certain limit procedure. In the next section the solution will be found directly without integration.

3. INITIAL PROBLEM

Let the single harmonic force $a_0 \exp(i\omega_0 t)$ is applied to the stationary point x_0, where $0 \leqslant x_0 \leqslant l$. Similar to equality (1), one can write

$$f[\partial^+ y(x_0, t)/\partial x - \partial^- y(x_0, t)/\partial x] + a_0 \exp(i\omega_0 t) = 0. \tag{4}$$

Equation (2) now reduces to the following:

$$\begin{aligned}\rho\partial^2 y(x, t)/\partial t^2 & - f\partial^2 y(x, t)/\partial x^2 \\ & + \sum_{n=-\infty}^{+\infty} [ky(nl, t) + k_1 \partial y(nl, t)/\partial t]\delta(x - nl) \\ & = a_0 \exp(i\omega_0 t)\delta(x - x_0).\end{aligned} \tag{5}$$

Denote dimensionless values by

$$X = x/l, \qquad X_0 = x_0/l, \qquad T = v_* t/l, \qquad Y(X, T) = y(x, t)/l,$$
$$K = kl/f, \qquad K_1 = k_1/(\rho f)^{1/2}, \qquad A_0 = a_0/f, \qquad \Omega_0 = \omega_0 l/v_*,$$

where $v_* = (f/\rho)^{1/2}$ is the speed of a free wave, propagating in a free string. So, l/v_* is the time as the free wave moves over the distance l and the corresponding value of the dimensionless time T is 1. Except T, all these dimensionless values were used in reference [4], where $T = v_0 t/l$ was adopted as the dimensionless time. Variable X denotes the dimensionless longitudinal co-ordinate. The dimensionless co-ordinate X_0, where $0 \leqslant X_0 \leqslant 1$, marks the excitation point. Introducing the dimensionless values in (1), (4) and (5) and taking into account that $\delta(x - x_0) = \delta(X - X_0)/l$, one obtains

$$\partial^+ Y(n, T)/\partial X - \partial^- Y(n, T)/\partial X = KY(n, T) + K_1 \partial Y(n, T)/\partial T \tag{6}$$

$$\partial^+ Y(X_0, T)/\partial X - \partial^- Y(X_0, T)/\partial X = A_0 \exp(i\Omega_0 T) = 0, \tag{7}$$

$$\begin{aligned}\partial^2 Y(X, T)/\partial T^2 & - \partial^2 Y(X, T)/\partial X^2 \\ & + \sum_{n=-\infty}^{+\infty} [KY(n, T) + K_1 \partial Y(n, T)/\partial T]\delta(X - n) \\ & = A_0 \exp(i\Omega_0 T)\delta(X - X_0).\end{aligned} \tag{8}$$

Here and below, upper indices " $\pm$ " have the same sense as before. Now, consider the steady state oscillations of the string, caused by the stationary harmonic force. Every point of the string performs harmonic oscillations with the excitation frequency. Therefore,

$$Y(X, T) = A_0 \exp(i\Omega_0 T) A(X, \Omega_0), \tag{9}$$

where $A(X, \Omega_0)$ is the complex amplitude of oscillations of the string arbitrary point X, related to $A_0 = 1$ and caused by the stationary harmonic force of the dimensionless angular velocity Ω_0, applied to the point X_0. By introducing (9) into (6)–(8) and cancelling $\exp(\mathrm{i}\Omega_0 T)$, yields

$$\mathrm{d}^+ A(n, \Omega_0)/\mathrm{d}X - \mathrm{d}^- A(n, \Omega_0)/\mathrm{d}X = K_0 A(n, \Omega_0), \tag{10}$$

$$\mathrm{d}^+ A(X_0, \Omega_0)/\mathrm{d}X - \mathrm{d}^- A(X_0, \Omega_0)/\mathrm{d}X + 1 = 0. \tag{11}$$

The complex value $K_0 = K + \mathrm{i}\Omega_0 K_1$ presents both the suspension dimensionless stiffness K and viscous damping K_1. Excluding the points X_0 and n, where the concentrated forces are applied, and considering any of segments between these points, one obtains

$$\mathrm{d}^2 A(X, \Omega_0)/\mathrm{d}X^2 + \Omega_0^2 A(X, \Omega_0) = 0. \tag{12}$$

Thus, the complex amplitude $A(X, \Omega_0)$ of the segment point is governed by the ordinary differential equation (12). Consider a boundary problem which connects this amplitude to the ones related to the beginning and end of the segment. Solving the equation over all the segments and sticking the solutions together by means of equalities (10) and (11), one can obtain the complex amplitude of the entire string. After this, the string transverse deflection can be obtained by means of equality (9).

Start with the loaded segment $0 \leqslant X \leqslant 1$. The excitation point X_0 divides this segment into to parts. Consider them separately.

If $0 \leqslant X \leqslant X_0$, then

$$\begin{aligned} A(X, \Omega_0) &= [A(0, \Omega_0)\sin(\Omega_0(X_0 - X)) \\ &\quad + A(X_0, \Omega_0)\sin(\Omega_0 X)]/\sin(\Omega_0 X_0), \\ \mathrm{d}^- A(X_0, \Omega_0)/\mathrm{d}X &= [A(X_0, \Omega_0)\cos(\Omega_0 X_0) \\ &\quad - A(0, \Omega_0)]\,\Omega_0/\sin(\Omega_0 X_0). \end{aligned} \tag{13}$$

If $X_0 \leqslant X \leqslant 1$, then

$$\begin{aligned} A(X, \Omega_0) &= [A(X_0, \Omega_0)\sin(\Omega_0(1 - X)) \\ &\quad + A(1, \Omega_0)\sin(\Omega_0(X - X_0))]/\sin(\Omega_0(1 - X_0)), \\ \mathrm{d}^+ A(X_0, \Omega_0)/\mathrm{d}X &= [A(1, \Omega_0) \\ &\quad - A(X_0, \Omega_0)]\cos(\Omega_0(1 - X_0))]/\sin(\Omega_0(1 - X_0)). \end{aligned} \tag{14}$$

Taking these and equality (11) into account, one obtains the equality

$$\begin{aligned} A(X_0, \Omega_0)\sin\Omega_0 &= \sin(\Omega_0(1 - X_0))\sin(\Omega_0 X_0)/\Omega_0 \\ &\quad + A(0, \Omega_0)\sin(\Omega_0(1 - X_0)) + A(1, \Omega_0)\sin(\Omega_0 X_0) \end{aligned} \tag{15}$$

which links the complex amplitudes that correspond to three points 0, X_0 and 1. Consider a set of the regular segments $0 \neq n \leqslant X \leqslant n+1$. For every such segment,

$$\begin{aligned} A(X, \Omega_0) &= [A(n, \Omega_0)\sin(\Omega_0(n+1-X)) \\ &\quad + A(n+1, \Omega_0)\sin(\Omega_0(X-n))]/\sin\Omega_0, \qquad (16) \\ \mathrm{d}^- A(n+1, \Omega_0)/\mathrm{d}X &= [A(n+1, \Omega_0)\cos\Omega_0 \\ &\quad - A(n, \Omega_0)]\Omega_0/\sin\Omega_0. \end{aligned}$$

Let $n > 0$ now. Substituting $n+1$ for n and X in the derivative to expression (16), one obtains

$$\mathrm{d}^+ A(n+1, \Omega_0)/\mathrm{d}X = [A(n+2, \Omega_0) - A(n+1, \Omega_0)\cos\Omega_0]\Omega_0/\sin\Omega_0.$$

Substituting $n+1$ for n in equality (10) as well, and, then, substituting into it both expressions, which follow expression (16), one yields the second order linear homogeneous difference equations [17, 18]

$$A(n+2, \Omega_0) - 2CA(n+1, \Omega_0) + A(n, \Omega_0) = 0$$

with constant coefficients, which links the values of the complex amplitude that correspond to three successive suspension points $n > 0$, $n+1$ and $n+2$. Elementary solutions to this equation can be found in the following form:

$$A(n+1, \Omega_0) = \Theta A(n, \Omega_0), \qquad (17)$$

where the constant Θ is unknown. Substituting this into the difference equation, one obtains a quadratic equation

$$\Theta^2 - 2C\Theta + 1 = 0$$

to determine Θ. Here and above

$$2C = 2\cos\Omega_0 + K_0 \sin\Omega_0/\Omega_0. \qquad (18)$$

The coefficient $2C$ depends on the dimensionless angular velocity Ω_0. Two roots $\Theta_{0,1}$ of the equation are always connected to each other with the equality $\Theta_0\Theta_1 = 1$ and so they can be equal to ± 1 only simultaneously. In both cases, $C = \pm 1$ correspondingly. If $K_1 \neq 0$ and so K_0 is not real, then these cases take place only with $\Omega_0 = \pi n$, $n \neq 0$. If viscous damping in the suspensions is absent, then $K_1 = 0$ and so $K_0 = K$ is real, as well as C. In the last case, there is an additional set of values Ω_0, which bring to $C = \pm 1$. All these values correspond to resonance or anti-resonance in the periodic string (see reference [4]). Thus, if $|C| > 1$, then roots

$$\Theta_{0,1} = C \pm \sqrt{C^2 - 1}$$

are different real numbers of the same sign. This means that one of the two values $|\Theta_{0,1}|$ is more than 1, but another one is less than 1. If $|C| < 1$, then the roots

$$\Theta_{0,1} = C \pm \mathrm{i}\sqrt{1 - C^2}$$

are complex conjugated numbers and $|\Theta_{0,1}| = 1$. The reverse statement takes place too. Thus, if $K_1 \neq 0$ and $\Omega_0 \neq \pi n$, except $n = 0$, then the quadratic equation has two different roots, which satisfy the following conditions: $|\Theta_0| > 1$ and $|\Theta_1| < 1$ that can be easily checked. In this case, the fundamental solution to the difference equation is a linear combination of two elementary ones written above. Taking into account that $A(n+1, \Omega_0)$ tends to zero as $n \to +\infty$, one should adopt for every $n > 0$

$$A(n+1, \Omega_0) = \Theta_1 A(n, \Omega_0) = \cdots = \Theta_1^n A(1, \Omega_0).$$

Now, expression (16) can be reduced to the next:

$$A(X, \Omega_0) = \Theta_1^{n-1} A(1, \Omega_0)[\sin(\Omega_0(n+1-X)) + \Theta_1 \sin(\Omega_0(X-n))]/\sin \Omega_0, \quad 0 < n \leqslant X \leqslant n+1. \tag{19}$$

Similarly, one can obtain $A(n, \Omega_0) = \Theta_0^n A(0, \Omega_0)$ for every $n < 0$ and so $A(n, \Omega_0)$ tends to zero again as $n \to -\infty$. Thus,

$$A(X, \Omega_0) = \Theta_0^n A(0, \Omega_0)[\sin(\Omega_0(n+1-X)) + \Theta_0 \sin(\Omega_0(X-n))]/\sin \Omega_0, \quad n \leqslant X \leqslant n+1 \leqslant 0. \tag{20}$$

The last two expressions include only two unknown values $A(0, \Omega_0)$ and $A(1, \Omega_0)$. Equality (15) links the unknowns with the third one $A(X_0, \Omega_0)$. In order to find two missing equalities, one should consider two segments, neighbouring to the loaded one. Substituting $n = 1$ into expression (19), then $X = 1$ into the derivatives to expressions (19) and (14), one obtains the following:

$$\mathrm{d}^+ A(1, \Omega_0)/\mathrm{d}X = A(1, \Omega_0)(\Theta_1 - \cos \Omega_0)\Omega_0/\sin \Omega_0,$$

$$\mathrm{d}^- A(1, \Omega_0)/\mathrm{d}X = [A(1, \Omega_0)\cos(\Omega_0(1-X_0)) - A(X_0, \Omega_0)]\Omega_0/\sin(\Omega_0(1-X_0)).$$

Substituting both derivatives and $n = 1$ into equality (10) and making some calculations, one yields

$$A(X_0, \Omega_0)\sin \Omega_0 = A(1, \Omega_0)[\sin(\Omega_0 X_0) + \Theta_0 \sin(\Omega_0(1-X_0))]. \tag{21}$$

Let $n = -1$ now. Substituting $X = 0$ into derivatives to the expressions (13) and (20), one obtains

$$\mathrm{d}^+ A(0, \Omega_0)/\mathrm{d}X = [A(X_0, \Omega_0) - A(0, \Omega_0)\cos(\Omega_0 X_0)]\,\Omega_0/\sin(\Omega_0 X_0),$$
$$\mathrm{d}^- A(0, \Omega_0)/\mathrm{d}X = A(0, \Omega_0)(\cos\Omega_0 - \Theta_1)\Omega_0/\sin\Omega_0.$$

Making former calculations once again, one yields

$$A(X_0, \Omega_0)\sin\Omega_0 = A(0, \Omega_0)[\sin(\Omega_0(1 - X_0)) + \Theta_0 \sin(\Omega_0 X_0)]. \tag{22}$$

Now, one has three linear equations (15), (21) and (22), which link three unknowns $A(0, \Omega_0)$, $A(X_0, \Omega_0)$ and $A(1, \Omega_0)$. Solving the equations together one finally determines these values as

$$A(X_0, \Omega_0) = \frac{[\sin(\Omega_0(1 - X_0)) + \Theta_0 \sin(\Omega_0 X_0)][\sin(\Omega_0(1 - X_0)) + \Theta_1 \sin(\Omega_0 X_0)]}{\Omega_0(\Theta_0 - \Theta_1)\sin\Omega_0}, \tag{23}$$

$$A(0, \Omega_0) = \frac{\sin(\Omega_0(1 - X_0)) + \Theta_1 \sin(\Omega_0 X_0)}{\Omega_0(\Theta_0 - \Theta_1)}, \tag{24}$$

$$A(1, \Omega_0) = \frac{\Theta_1 \sin(\Omega_0(1 - X_0)) + \sin(\Omega_0 X_0)}{\Omega_0(\Theta_0 - \Theta_1)} \tag{25}$$

and completes solution of the initial problem. After substituting these values into expressions (13), (14), (19) and (20), one can calculate the complex amplitude ($A(X, \Omega_0)$) of the steady state oscillations, related to an arbitrary point of the periodic string. This calculation needs no integration, but includes evaluation of the complex value $\Theta_0 - \Theta_1$, which depends on Ω_0 via C, and according to the quadratic equation, can be reduced to the following expression:

$$\Theta_0 - \Theta_1 = \pm 2\sqrt{C^2 - 1}. \tag{26}$$

The last shows that this value represents an infinitesimal value of the 1/2nd order as $C \to \pm 1$. The value $A(X, \Omega_0)$ can approach infinity in the presence of the suspension viscous damping. The right sign in the right-hand side of equality (26) can be chosen, if one takes into account that $|\Theta_0| > 1$ and $|\Theta_1| < 1$. This can be done by means of a numerical procedure. If $|A(X, \Omega_0)|$ is calculated, then this sign plays no role and $K_1 = 0$ (no resistance) can be adopted. Calculations of $|A(0, \Omega_0)|$ with $X_0 = 0$ and $K_1 = 0$ for different values of the suspension dimensionless stiffness K and the excitation dimensionless angular velocity Ω_0 are shown in Figure 2. There is anti-resonance, which corresponds to $\Omega_0 = \pi n$, $n \neq 0$, n is an integer, and does not depend on K. The explanation of this can be found in reference [4]. As K increases, the resonance angular velocity increases too. Similar calculations were performed in reference [4] by means of integration, which is possible if $K_1 \neq 0$ only.

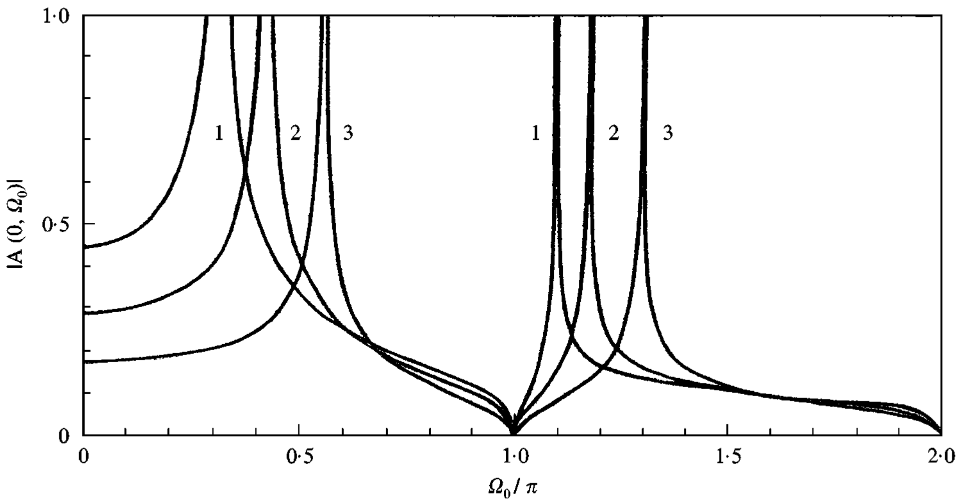

Figure 2. Frequency response to stationary excitation; 1, $K = 1$; 2, $K = 2$; 3, $K = 4$.

If one defines the sine functions in expressions (13), (14) and (16) in terms of exponential functions and after that substitutes them into expression (9), then the solution to the initial problem can be presented as a sum of a direct wave $\exp(\mathrm{i}\Omega_0(X - T))$ and a reverse wave $\exp(\mathrm{i}\Omega_0(X + T))$, which propagate over a span of the string with the same speed in opposite directions. Each suspension splits any of these waves into direct and reverse waves again, causing damping of oscillations far away from the excitation point. This creates a stationary system of such waves over the string. The modulus and phase of the complex number Θ in equality (17) represent this damping and the phase lag over a span as well as over any string segment of the same length. The last follows from expression (16). The phase lag defines the phase velocity, which depends on K and Ω_0.

The periodic string static deflection can be calculated by means of the same formulae, if one takes into account that $\sin(\Omega_0 X)/\Omega_0$ tends to X as Ω_0 tends to zero. This means that all spans of the periodic string reduce to straight lines. The number Θ in equality (17), which is real and positive now, indicates the string deflection damping. In the static case,

$$\Theta_1 = 2/(K + 2 + \sqrt{K^2 + 4K}), \qquad \Theta_0 - \Theta_1 = 2\sqrt{K^2 + 4K}.$$

If K increases, then $\Theta_0 - \Theta_1$ increases too, but Θ_1 tends to zero. The former means that the string deflection becomes less, if K increases. The latter means that the string deflection vanishes faster as $X \to \pm\infty$. Calculation of the string static deflection, which is shown in Figure 3, confirms this.

4. RESPONSE TO AN IMPACT

Consider the following transverse load:

$$q(x, t) = b_0 \delta(t - t_0)\delta(x - x_0), \tag{27}$$

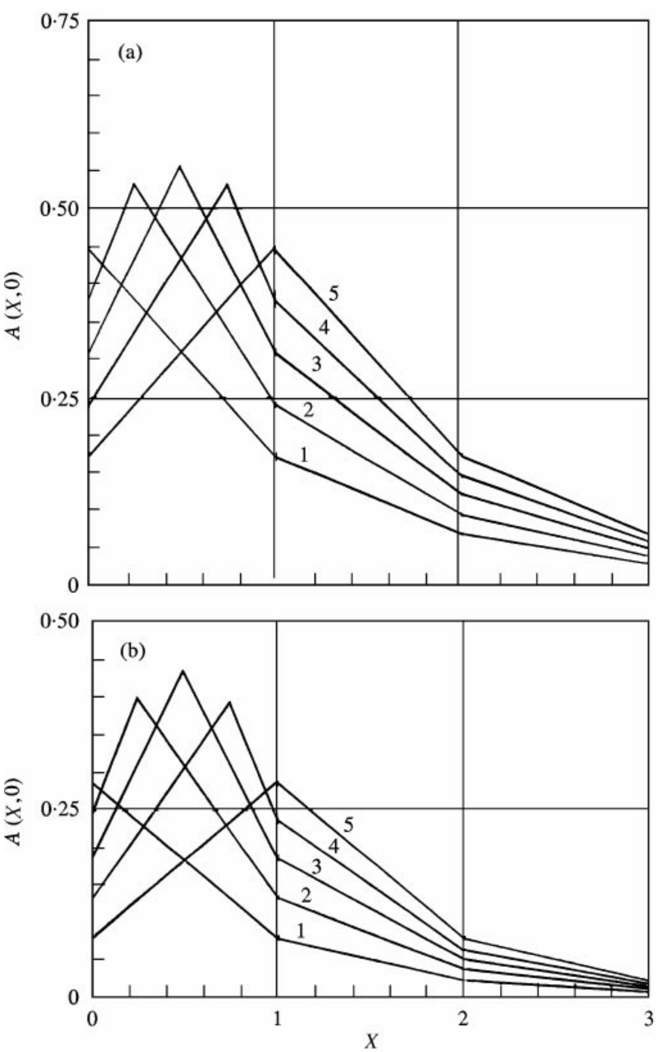

Figure 3. Static deflection; (a) $K = 1$; (b) $K = 2$; 1, $X_0 = 0$; 2, $X_0 = 0{\cdot}25$; 3, $X_0 = 0{\cdot}5$; 4, $X_0 = 0{\cdot}75$; 5, $X_0 = 1$.

which consists of a double Dirac function and describes an instantaneous impulsive force with the magnitude b_0, applied to the string point x_0 at the time t_0. The Dirac function $\delta(t - t_0)$ can be used to present action of a very large force over a very small time interval. On the other hand, a concentrated transverse load can be presented as a sequence of such impulses, which act independent of each other due to linearity of the considered structure. Substituting the load (27) into equation (2), then introducing dimensionless values and taking into account that $\delta(t - t_0) = v_* \delta(T - T_0)/l$ where $T_0 = v_* t_0/l$, yields

$$\partial^2 Y(X, T)/\partial T^2 - \partial^2 Y(X, T)/\partial X^2$$

$$+\sum_{n=-\infty}^{+\infty}[KY(n,T)+K_1\partial Y(n,T)/\partial T]\delta(X-n)$$
$$=B_0\delta(T-T_0)\delta(X-X_0),\qquad B_0=b_0/(l(\rho f)^{1/2}). \tag{28}$$

The functional-differential equation (28) describes the string response to the single instantaneous impulse of force and may be directly solved by means of double integration. In order to avoid a risky double integration of the double Dirac function in the right-hand side of this equation, the Dirac function $\delta(T-T_0)$ can be presented by means of the Fourier integral in the following form:

$$\delta(T-T_0)=(2\pi)^{-1}\int_{-\infty}^{+\infty}\exp(\mathrm{i}\Omega_0(T-T_0))\,\mathrm{d}\Omega_0.$$

The integral shows that instantaneous impulsive force is an infinite sum of harmonic forces with the same amplitude and phase, whose angular velocities fill the infinite band. This integral does not converge in the usual mathematical sense and should be used with caution. Introducing this integral into the right-hand side of equation (28), yields

$$B_0(2\pi)^{-1}\int_{-\infty}^{+\infty}\exp(\mathrm{i}\Omega_0(T-T_0))\delta(X-X_0)\mathrm{d}\Omega_0.$$

Taking into account that the last integrand coincides with the right-hand side of equation (8), one concludes that the solution to equation (28) can be obtained by a single integration of solution (9) to equation (8) after substituting $B_0(2\pi)^{-1}$ for A_0. Thus, the following improper single integral

$$Y(X,T)=B_0(2\pi)^{-1}\int_{-\infty}^{+\infty}\exp(\mathrm{i}\Omega(T-T_0))A(X,\Omega)\,\mathrm{d}\Omega \tag{29}$$

represents the solution to equation (28). The integration variable Ω is used instead Ω_0. Integrand (29) may approach infinity in the presence of the suspension viscous damping (see reference [4] and the previous section), but stays integrable.

Consider, for example, the response of a periodic string to an impulsive force, applied to the suspension point $X_0=0$ at the time $T_0=0$. This response at the same suspension point and at the neighbour one is represented by two following integrals:

$$Y(0,T)=\frac{B_0}{2\pi}\int_{-\infty}^{+\infty}\frac{\exp(\mathrm{i}\Omega T)\sin\Omega\,\mathrm{d}\Omega}{\Omega(\Theta_0-\Theta_1)},$$

$$Y(1,T)=\frac{B_0}{2\pi}\int_{-\infty}^{+\infty}\frac{\Theta_1\exp(\mathrm{i}\Omega T)\sin\Omega\,\mathrm{d}\Omega}{\Omega(\Theta_0-\Theta_1)}.$$

The last two integrands include complex values $\exp(\mathrm{i}\Omega T)$, Θ_1 and $\Theta_0-\Theta_1$. The real and imaginary parts of the first one are easily separated from each other. But the second and the third ones split into their real and imaginary parts numerically at every integration step. This allows one to calculate both real and imaginary parts of the integrals only numerically. The imaginary parts of both integrands are odd function of the integration variable Ω and so both integrals are real. Results of the evaluation of these integrals are shown in Figure 4. Curve 1 shows the string response to an impulse at the suspension point. Curve 2 shows this response at the neighbouring suspension.

Some interesting details in the curves should be mentioned and discussed. One sees an instantaneous lifting of the excitation point at the time $T = 0$, when the excitation takes place. If an impulse of force has been applied to a concentrated mass, which was at rest, then this mass gains a certain finite speed. The string mass is distributed along the string and the string disturbance spreads with a finite speed. This means that the impulse of force, applied to the string point with zero mass, cause an infinite speed of this point, while all other string points are still at rest. The suspension viscous resistance cannot prevent the infinite speed because it is caused by the infinite force. Therefore, the instantaneous lifting of the excitation point seems to be justified in such circumstances. Small fluctuations in this curve just before and after the excitation are caused by calculation errors. They quickly and randomly change with any change in the numerical integration step and interval.

The impulsive force causes two shock waves, which propagate over string spans with the same speed in opposite directions without distortion. Both waves reach two neighbour suspensions and cause their deflection at the dimensionless time $T = 1$. The instantaneous lifting in curve 2 at the time marks this event. Small fluctuations accompany this lifting too. Each neighbour suspension splits the wave into direct and reverse ones. Both reverse waves reach the excitation point at the dimensionless time $T = 2$. This event is marked with an instantaneous drop in curve 1 at the time. Further splits of the direct and reverse waves by these and all the following suspensions lead to multiple instantaneous changes in both curves, related to integer values of T. Thus, curves 1 and 2 shows that solution (29) rightly presents the string behaviour [13] and periodic nature.

5. RESPONSE TO SUDDEN APPLICATION OF A FORCE

Consider the transverse load (3), applied to the periodic string again. Now suppose that the concentrated harmonic force $a_0 \exp(i\omega_0 t)$ suddenly appears at the time $t = 0$ and the periodic string has been at rest before the appearence of the excitation. Over the small time interval $0 \leqslant t_0 \leqslant t \leqslant t_0 + dt_0$, the string experiences action of the impulse of force with the magnitude $a_0 \exp(i\omega_0 t_0)\, dt_0 = b_0 \exp(i\Omega_0 T_0)\, dT_0$, where $b_0 = a_0 l(\rho/f)^{1/2}$, and so $B_0 = a_0/f$. Substituting this impulse into integral (29), one obtains the string response to the impulse in the following form:

$$a_0/(2\pi f)\exp(i\Omega_0 T_0)\, dT_0 \int_{-\infty}^{+\infty} \exp(i\Omega(T - T_0)) A(X, \Omega)\, d\Omega.$$

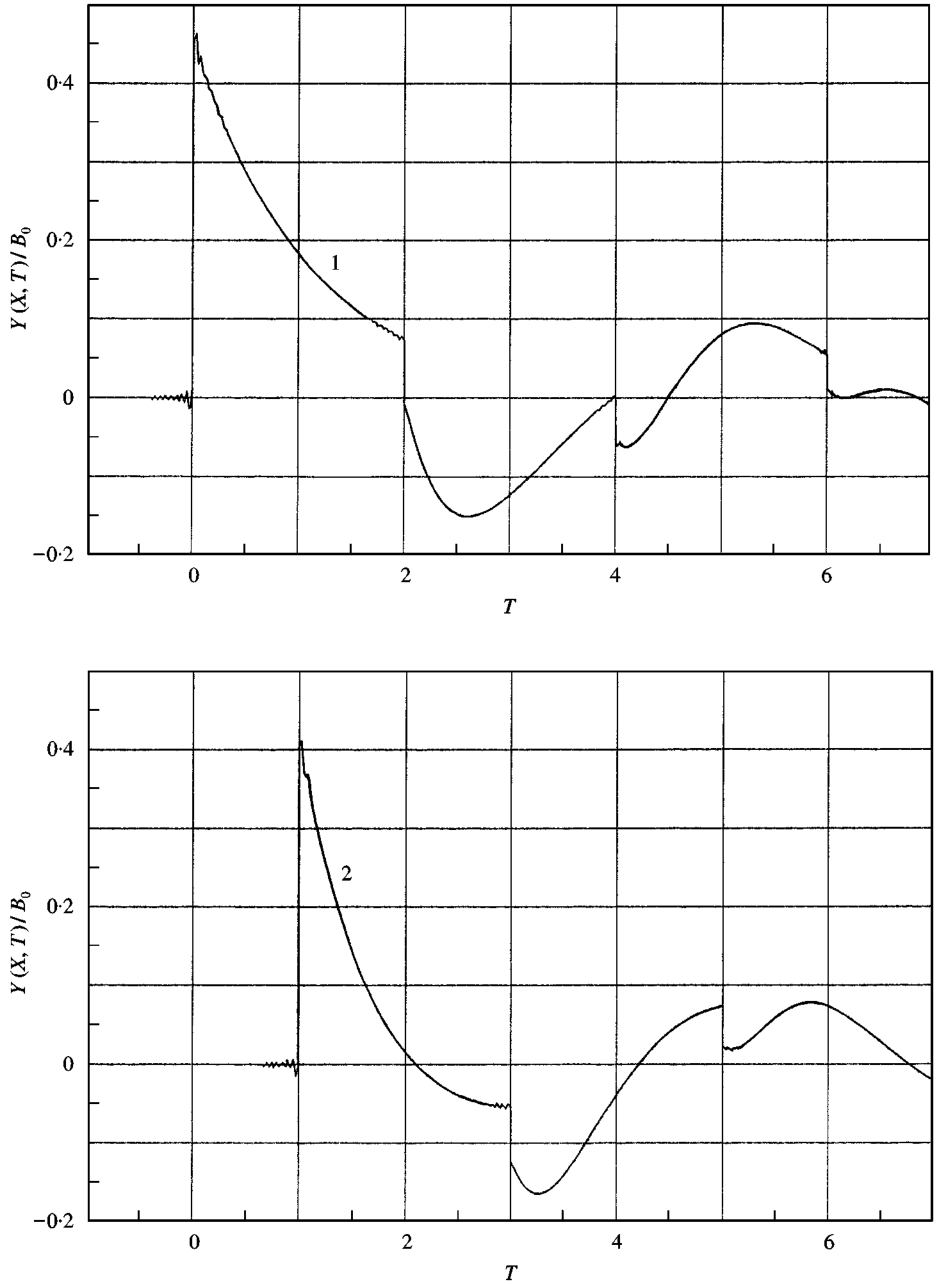

Figure 4. Response to an impact; $K = 2$, $K_1 = 0{\cdot}1$, $X_0 = 0$; 1, $X = 0$; 2, $X = 1$.

The action of this force can be considered as the action of the set of consequtive instantaneous impulses. After integration the response with respect to T_0 over the time segment $0 \leqslant T_0 \leqslant T$, yields

$$Y(X, T) = a_0/(2\pi f)\int_0^T \exp(\mathrm{i}\Omega_0 T_0)\,\mathrm{d}T_0 \int_{-\infty}^{+\infty} \exp(\mathrm{i}\Omega(T - T_0))A(X, \Omega)\,\mathrm{d}\Omega.$$

This is the string response at the time T to action of the concentrated harmonic force over the time interval mentioned. In order to escape double integration, one should change the sequence of integration and calculate the inner integral. After this, the string response obtains the form of a single improper integral

$$Y(X, T) = \frac{a_0}{2\pi i f}\int_{-\infty}^{+\infty} \frac{\exp(i\Omega_0 T) - \exp(i\Omega T)}{\Omega_0 - \Omega} A(X, \Omega)\, d\Omega. \tag{30}$$

This integral represents a transient system of waves, which turns into the stationary one as $T \to +\infty$. Consider the periodic string unsteady response at the suspension point $X = 0$ to the harmonic force, suddenly applied to the same point. Substituting $A(0, \Omega)$ into integral (30) one obtains

$$Y(0, T) = \frac{a_0}{2\pi i f}\int_{-\infty}^{+\infty} \frac{\exp(i\Omega_0 T) - \exp(i\Omega T)}{\Omega_0 - \Omega} \frac{\sin\Omega\, d\Omega}{\Omega(\Theta_0 - \Theta_1)}. \tag{31}$$

One can calculate both the real and imaginary parts of integral (31) in the same manner as before. Figure 5 shows the complex value (31) dependence on the dimensionless time T in three-dimensional space. The horizontal axis OR is real, while the vertical one OI is imaginary. There are right angles between these the T-axis. Calculations correspond to $K = 2$ and $K_1 = 0{\cdot}1$. Curves 1, 2 and 3 relate to the following values $\pi/2$, π and $0{\cdot}451\pi$ of the dimensionless angular velocity Ω_0. These values correspond to a regular excitation, anti-resonance and resonance (see Figure 2). Therefore, curve 1 gradually turns into a regular spiral that corresponds to the steady state oscillations. Curve 2 is more complex. It gradually developes just after the harmonic force appearence. Then, it shrinks like the *dying swan*, curling up the T-axis. Due to resistance in the suspensions, a certain distance between the curve and the T-axis remains (see reference [4]). Curve 3 moves off the T-axis as $T \to +\infty$.

If $\Omega_0 = 0$, then the suddenly applied force is a constant one of magnitude a_0. In this case, integral (30) reduces to the following one:

$$Y(X, T) = \frac{a_0}{2\pi i f}\int_{-\infty}^{+\infty} \frac{\exp(i\Omega T) - 1}{\Omega} A(X, \Omega)\, d\Omega. \tag{32}$$

The imaginary part of integral (32) becomes zero. In the particular cases $X_0 = 0$ and $X = 0$ or 1, the real integral (32) reduces to

$$Y(0, T) = \frac{a_0}{2\pi i f}\int_{-\infty}^{+\infty} \frac{\exp(i\Omega T) - 1}{\Omega^2} \frac{\sin\Omega\, d\Omega}{\Theta_0 - \Theta_1},$$

$$Y(1, T) = \frac{a_0}{2\pi i f}\int_{-\infty}^{+\infty} \frac{\exp(i\Omega T) - 1}{\Omega^2} \frac{\Theta_1 \sin\Omega\, d\Omega}{\Theta_0 - \Theta_1}.$$

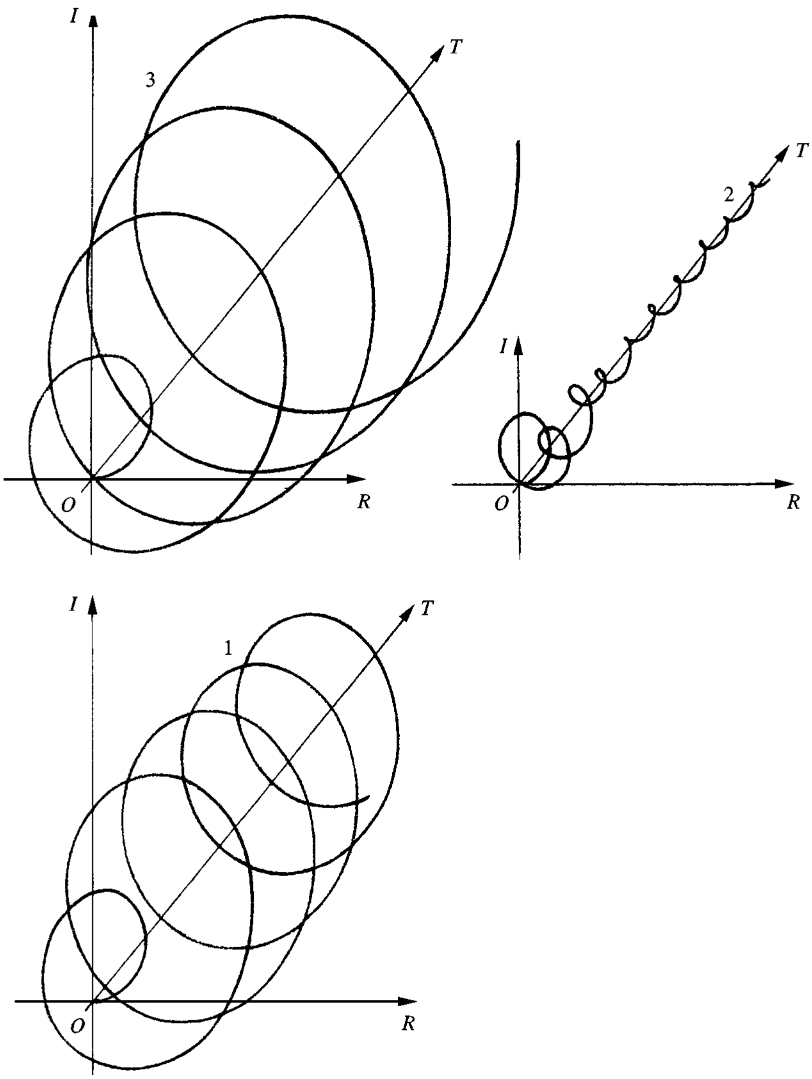

Figure 5. Response to a suddenly applied harmonic force; $K = 2$, $K_1 = 0{\cdot}1$; 1, $\Omega_0 = \pi/2$; 2, $\Omega_0 = \pi$; 3, $\Omega_0 = 0{\cdot}415\pi$.

Figure 6 shows results of calculation of these integrals with $K = 2$ and $K_1 = 0{\cdot}1$ again. Curve 1 indicates the suspension point response to the constant force, suddenly applied to the same point, while curve 2 shows the neighbour suspension point response to the same force. Two dotted lines show the static deflection, whose values relate to the points and has been taken from Figure 3. Curve 1 begins to raise gradually just after the constant force appearance at the time $T = 0$ and Curve 2 does at the time $T = 1$, when the string disturbance reaches the point $X = 1$. After that, both curves oscillate around the dotted lines with the dominant dimensionless angular velocity of $0{\cdot}415\pi$, that corresponds to the first resonance (see Figure 2). The oscillations vanish as $T \to +\infty$. There is no instantaneous lifting in the curves as well as small fluctuations. This follows the smoothing effect of the integration with respect to T.

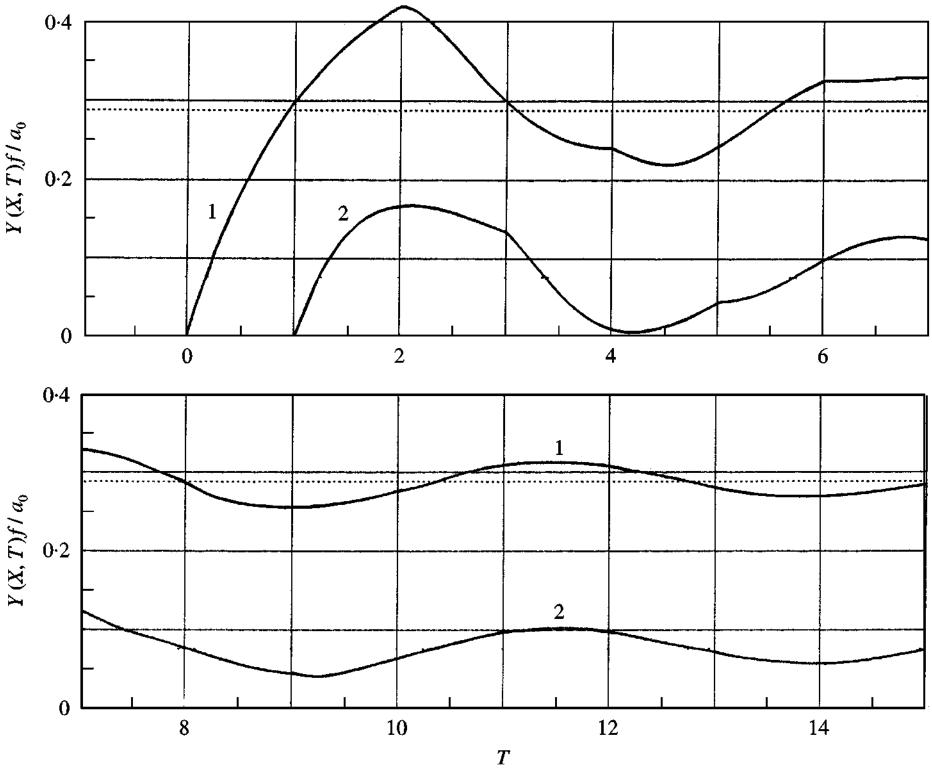

Figure 6. Response to a suddenly applied constant force; $K = 2$, $K_1 = 0{\cdot}1$, $X_0 = 0$; 1, $X = 0$; 2, $X = 1$.

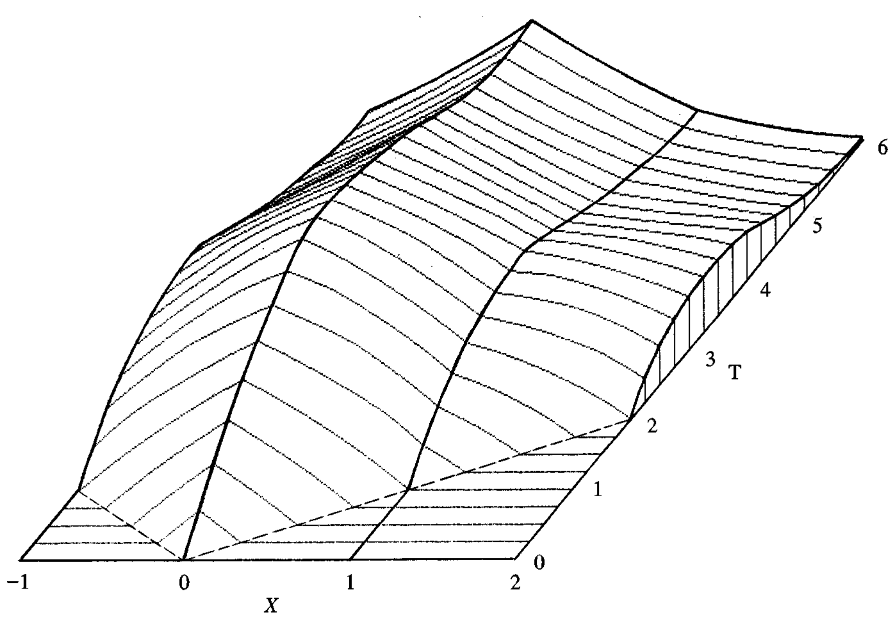

Figure 7. Periodic string unsteady deflection, caused by a suddenly applied constant force. Transition to a static deflection; $K = 1$, $K_1 = 0{\cdot}15$.

Figure 7 shows oscillations of the string segment, which consists of three spans. Calculations have been made by means of integral (32) for $X_0 = 0$ with $K = 1$ and $K_1 = 0{\cdot}15$. One sees four curves, which stretch in the direction of the T-axis and indicate the deflection of the suspension points at X equals -1, 0 (the excitation point), 1 and 2. These four curves are crossed by 25 curves, which successively presents the string shape at the dimensionless time that ranges from $T = 0$ to 6 with the time interval 0·25. The front one, that relates to $T = 0$, is straight. One sees fractures in other ones that are caused by the concentrated forces, applied to the string at the suspension points. There are two fractures in the string shape that follow the string dynamical behaviour itself [13]. The string disturbance propagation with the constant speed is seen just after the appearance of the constant force at $T = 0$. Two oblique dotted lines mark boundaries of the string disturbance and those fractures. As time T increases, formation of the string static shape, shown in Figure 3, is clearly seen.

6. CONCLUSIONS

The unsteady response of an infinite periodic structure to a forced excitation has been calculated. The infinite structure considered is a stretched string, supported by equidistantly spaced identical suspensions. Each suspensions consists of a spring and a dashpot with viscous damping, in parallel. Any forced excitation represents a sequence or a distribution of impluses. The periodic string response to an instantaneous impulse of force, applied to the structure at an arbitrary point, has been found and presented in the form of an improper single integral. This integral has been used to calculate the strings response to the suddenly applied harmonic or constant force. The calculated response rightly represents the string behaviour and periodic nature.

ACKNOWLEDGMENTS

The author is grateful to the referee, who has called the author's attention to the propagation aspect of the solution to the stationary problem. As a result, an interesting link between the mathematical formalism and wave propagation has been cleared up.

This study was partly supported by the Fund for Fundamental Studies of the Ministry of Railway Transport of the RF (grant NIOKR 47/98).

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